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The Dirac equation with a Coulomb potential in *D* dimensions*

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Abstract

The Dirac equation is generalized to $D+1$ spacetime case. The radial equations of this quantum system are obtained and solved exactly by the Tricomi equation approach. The energy levels $E(n, l, D)$ are analytically presented. The dependences of the energy $E(n, l, D)$ on the dimension D are also analysed. It is shown that the energy $E(n, l, D)$ $(l \neq 0)$ is almost independent of the quantum number l, while $E(n, 0, D)$ first decreases and then increases as the dimension D increases. Unexpectedly, there is an absence of bound states for this quantum system with the special case $D = 1$, which is explained in some detail from the fact that the eigenvalues and eigenfunctions do not exist for $D = 1$.

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1. Introduction

The exact solutions of the non-relativistic and relativistic equations with the Coulomb potential play an important role in quantum mechanics [1–3]. With the interest in the higher dimensional field theory, the D-dimensional Schrödinger equation with the Coulomb potential has been studied by many authors [4–11]. The solutions of the Dirac equation with the Coulomb potential have also been studied in the usual 3D [12–16], 2D [17] and 1D [18] space. Moreover, the qualitative properties of the Dirac particle in a central potential and the discrete eigenvalues of the radial Dirac operator have been discussed [19, 20]. The Klein–Gordon equation with the Coulomb potential in $D+1$ dimensions has been discussed by several authors [21–23]. To fill in the gap between the Schrödinger equation case and the Klein–Gordon one, we have studied the Dirac equation with the Coulomb potential in $D+1$ dimensions recently [24], where we have used a great deal of space to study how to generalize the Dirac equation to the

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$D+1$ spacetime case and obtained the radial wavefunctions of the system and also obtained the exact solutions of radial wavefunctions in a very small space by the power series approach since such a method has been used to study the Dirac equation with the Coulomb potential in two dimensions in some detail [17]. The present work is twofold. The first purpose is to study the exact solutions of the quantum system by the Tricomi equation approach, which is different from the power series one as done in [17]. The second one, which is the main purpose of this work, is to investigate the dependences of the energy levels $E(n, l, D)$ on the dimension D if the dimension D is considered as a *continuous* variable, as proposed by Nieto [21]. It is noteworthy that 'the Coulomb potential' in these works, including the present paper, means a potential which is inversely proportional to r.

This paper is organized as follows. Section 2 is devoted to the generalization of the Dirac equation to the $D+1$ spacetime case. The radial wavefunctions of Dirac equation in $D+1$ dimensions are briefly introduced. By the Tricomi equation approach, the exact solutions of the radial equations of this quantum system are obtained in section 3. In section 4, the dependences of the energy levels $E(n, l, D)$ on the dimension D are analysed in some detail. Section 5 is devoted to explaining the reason why there is an absence of bound states for $D = 1$. Some concluding remarks are given in section 6.

2. Dirac equation in $D+1$ dimensions

The Dirac equation in $D+1$ dimensions can be expressed as [25]

$$\mathrm{i}\sum_{\mu=0}^{D}\gamma^{\mu}(\partial_{\mu}+\mathrm{i}eA_{\mu})\Psi(\mathbf{x},t)=M\Psi(\mathbf{x},t) \tag{1}$$

where M is the mass of the particle, and $(D+1)$ matrices γ_{μ} satisfy the anti-commutative relations:

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\eta^{\mu\nu}\mathbf{1} \tag{2}$$

with the metric tensor $\eta^{\mu\nu}$ satisfying

$$\eta^{\mu\nu}=\eta_{\mu\nu}=\begin{cases}\delta_{\mu\nu} & \text{when} \quad \mu=0\\ -\delta_{\mu\nu} & \text{when} \quad \mu\neq 0.\end{cases} \tag{3}$$

Throughout this paper, the natural units $\hbar = c = 1$ are employed if not explicitly stated otherwise. Discussing the special case, where only the zero component of A_{μ} is non-vanishing and spherically symmetric

$$eA_0=V(r) \qquad A_a=0 \qquad \text{when} \quad a\neq 0. \tag{4}$$

The Hamiltonian $H(\mathbf{x})$ of the system is expressed as

$$\begin{aligned}&\mathrm{i}\partial_0\Psi(\mathbf{x},t)=H(\mathbf{x})\Psi(\mathbf{x},t) \qquad H(\mathbf{x})=\sum_{a=1}^{D}\gamma^0\gamma^a p_a+V(r)+\gamma^0 M\\ &p_a=-\mathrm{i}\partial_a=-\mathrm{i}\frac{\partial}{\partial x^a} \qquad 1\leqslant a\leqslant D.\end{aligned} \tag{5}$$

The orbital angular momentum operators L_{ab}, the spinor operators S_{ab} and the total angular momentum operators J_{ab} are defined as follows:

$$\begin{aligned}&L_{ab}=-L_{ba}=\mathrm{i}x_a\partial_b-\mathrm{i}x_b\partial_a \qquad S_{ab}=-S_{ba}=\mathrm{i}\gamma_a\gamma_b/2 \qquad J_{ab}=L_{ab}+S_{ab}\\ &J^2=\sum_{a<b=2}^{D}J_{ab}^2 \qquad L^2=\sum_{a<b=2}^{D}L_{ab}^2 \qquad S^2=\sum_{a<b=2}^{D}S_{ab}^2\end{aligned} \tag{6}$$

with $1 \leqslant a < b \leqslant D$. The eigenvalue of J^2 (L^2 or S^2) is denoted by the Casimir $C_2(\mathbf{M})$, where $\mathbf{M}$ is the highest weight of the representation to which the total (orbital or spinor) wavefunction belongs. The Casimir $C_2(\mathbf{M})$ can be calculated by the formula (e.g., see (1.131) in [26])

$$C_2(\mathbf{M}) = \mathbf{M}\cdot(\mathbf{M}+2\varrho) = \sum_{\mu,\nu=1}^{N} M_\mu d_\mu (A^{-1})_{\mu\nu}(M_\nu + 2) \tag{7}$$

where ϱ is the half sum of the positive roots in the Lie algebra, A^{-1} is the inverse of the Carton matrix and d_μ are the half-square lengths of the simple roots.

It is easy to show by the standard method [25] that J_{ab} and κ are commutant with the Hamiltonian $H(\mathbf{x})$,

$$\kappa = \gamma^0 \left\{ \sum_{a<b} \mathrm{i}\gamma^a\gamma^b L_{ab} + (D-1)/2 \right\} = \gamma^0\{J^2 - L^2 - S^2 + (D-1)/2\}. \tag{8}$$

Since the potential $V(r)$ is spherically symmetric, the symmetry group of the system is the SO(D) group. The hyperspherical coordinates in the real D-dimensional space have been introduced by some authors [27–29]

$$\begin{aligned} x^1 &= r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{D-1} \\ x^2 &= r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{D-1} \\ x^b &= r\cos\theta_{b-1}\sin\theta_k \cdots \sin\theta_{D-1} \qquad 3 \leqslant b \leqslant D-1 \\ x^D &= r\cos\theta_{D-1} \qquad \sum_{a=1}^{D}(x^a)^2 = r^2. \end{aligned} \tag{9}$$

The unit vector along $\mathbf{x}$ is usually denoted by $\hat{\mathbf{x}} = \mathbf{x}/r$. The volume element of the configuration space is

$$\begin{aligned} &\prod_{a=1}^{D} \mathrm{d}x^a = r^{D-1}\,\mathrm{d}r\,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \prod_{a=1}^{D-1}(\sin\theta_a)^{a-1}\,\mathrm{d}\theta_a \\ &0 \leqslant r \leqslant \infty \quad -\pi \leqslant \theta_1 \leqslant \pi \quad 0 \leqslant \theta_c \leqslant \pi \quad 2 \leqslant c \leqslant D-1. \end{aligned} \tag{10}$$

For simplicity, according to the representation theory of the SO(D) group [24, 26, 30, 31], we can obtain the radial wavefunctions of the D-dimensional Dirac equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}G_{KE}(r) + \frac{K}{r}G_{KE}(r) = [E - V(r) - M]F_{KE}(r) \\ &-\frac{\mathrm{d}}{\mathrm{d}r}F_{KE}(r) + \frac{K}{r}F_{KE}(r) = [E - V(r) + M]G_{KE}(r) \end{aligned} \tag{11}$$

with

$$K = \pm(2l + D - 1)/2. \tag{12}$$

The $G_{KE}(r)$ and $F_{KE}(r)$ are the radial functions of the system after separating the angle variables from the wavefunctions. The detailed information can be referred to [24, 26, 30, 31].

3. Solutions of radial equations in $D+1$ dimensions

In this section, we will study the solutions of radial equations (11) with the Tricomi equation approach, which is another approach to investigating the solutions in comparison with the power series approach used in our previous work [17, 24].

We now consider the Dirac equation with the Coulomb potential, namely,

$$V(r) = -\frac{\xi}{r} \tag{13}$$

where we should address the constant $\xi = Z\alpha > 0$ with the fine structure constant $\alpha = 1/137$, which is correct only in three dimensions.

Substitution of equation (13) into equation (11) allows us to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}G_{KE}(r) + \frac{K}{r}G_{KE}(r) &= \left(E - M + \frac{\xi}{r}\right)F_{KE}(r) \\ -\frac{\mathrm{d}}{\mathrm{d}r}F_{KE}(r) + \frac{K}{r}F_{KE}(r) &= \left(E + M + \frac{\xi}{r}\right)G_{KE}(r). \end{aligned} \tag{14}$$

It is found that solutions for the repulsive potential can be obtained from those for the attractive potential by interchanging $F_{KE} \leftrightarrow G_{-K-E}$ and $V(r) \leftrightarrow -V(r)$. Therefore, we only need to discuss the attractive potential, say $\xi > 0$. On the other hand, from the Sturm–Liouville theorem [32], there exist bound states with energy less than and near M for the attractive Coulomb potential and with the energy larger than and near $-M$ for the repulsive potential, if the interaction is not too strong.

Before further proceeding to do so, it is convenient to introduce ρ for the bound states

$$\rho = 2r\sqrt{M^2 - E^2} \qquad |E| < M. \tag{15}$$

We thus have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\rho}G_{KE}(\rho) + \frac{K}{\rho}G_{KE}(\rho) &= \left(-\frac{1}{2}\sqrt{\frac{M-E}{M+E}} + \frac{\xi}{\rho}\right)F_{KE}(\rho) \\ \frac{\mathrm{d}}{\mathrm{d}\rho}F_{KE}(\rho) - \frac{K}{\rho}F_{KE}(\rho) &= \left(-\frac{1}{2}\sqrt{\frac{M+E}{M-E}} - \frac{\xi}{\rho}\right)G_{KE}(\rho). \end{aligned} \tag{16}$$

Define the wavefunctions $\Phi_{\pm}(\rho)$ with the forms

$$G_{KE}(\rho) = \sqrt{M-E}[\Phi_+(\rho) + \Phi_-(\rho)] \qquad F_{KE}(\rho) = \sqrt{M+E}[\Phi_+(\rho) - \Phi_-(\rho)]. \tag{17}$$

Substitution of equation (17) into equation (16) allows us to write down

$$\begin{aligned} [\Phi'_+(\rho) + \Phi'_-(\rho)] + \frac{K}{\rho}[\Phi_+(\rho) + \Phi_-(\rho)] &= \left[-\frac{1}{2} + \frac{\xi}{\rho}\sqrt{\frac{M+E}{M-E}}\right][\Phi_+(\rho) - \Phi_-(\rho)] \\ [\Phi'_+(\rho) - \Phi'_-(\rho)] - \frac{K}{\rho}[\Phi_+(\rho) - \Phi_-(\rho)] &= \left[-\frac{1}{2} - \frac{\xi}{\rho}\sqrt{\frac{M-E}{M+E}}\right][\Phi_+(\rho) + \Phi_-(\rho)]. \end{aligned} \tag{18}$$

where, and hereafter, $\Phi'_+(\rho)$ and $\Phi'_-(\rho)$ denote the first derivative with respect ρ.
Their addition and subtraction lead to

$$\begin{aligned} \Phi'_+(\rho) - \left(\frac{\xi E}{\rho\sqrt{M^2-E^2}} - \frac{1}{2}\right)\Phi_+(\rho) &= -\left(\frac{K}{\rho} + \frac{\xi M}{\rho\sqrt{M^2-E^2}}\right)\Phi_-(\rho) \\ \Phi'_+(\rho) + \left(\frac{\xi E}{\rho\sqrt{M^2-E^2}} - \frac{1}{2}\right)\Phi_-(\rho) &= -\left(\frac{K}{\rho} - \frac{\xi M}{\rho\sqrt{M^2-E^2}}\right)\Phi_+(\rho). \end{aligned} \tag{19}$$

Taking the following conventions

$$\tau = \frac{\xi E}{\sqrt{M^2-E^2}} \qquad \tau' = \frac{\xi M}{\sqrt{M^2-E^2}} \tag{20}$$

we have

$$\begin{aligned}\Phi'_+(\rho) - \left(\frac{\tau}{\rho} - \frac{1}{2}\right)\Phi_+(\rho) &= -\frac{\tau' + K}{\rho}\Phi_-(\rho)\\ \Phi'_-(\rho) + \left(\frac{\tau}{\rho} - \frac{1}{2}\right)\Phi_-(\rho) &= \frac{\tau' + K}{\rho}\Phi_+(\rho)\end{aligned} \tag{21}$$

from which we can obtain the following important second-order differential equations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \left(-\frac{1}{4} + \frac{\tau \pm 1/2}{\rho} - \frac{\eta^2}{\rho^2}\right)\right]\Phi_\pm(\rho) = 0 \qquad \eta^2 = K^2 - \xi^2. \tag{22}$$

For the weak Coulomb potential, we have

$$\eta = \sqrt{K^2 - \xi^2} > 0. \tag{23}$$

It is found that equation (22) is a special case of the Tricomi equation [33], which can be expressed as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(a + \frac{b}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\alpha + \frac{\beta}{x} + \frac{\xi}{x^2}\right)y = 0. \tag{24}$$

From the behaviour of the wavefunctions at the origin and at infinity, we define

$$\Phi_\pm(\rho) = \rho^n\,\mathrm{e}^{-\rho/2}R_\pm(\rho). \tag{25}$$

Substitution of this into (22) leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}R_\pm(\rho) + \left(-1 + \frac{1 + 2\eta}{\rho}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}R_\pm(\rho) + \frac{\tau - \eta - 1/2 \pm 1/2}{\rho}R_\pm(\rho) = 0 \tag{26}$$

whose solutions are the confluent hypergeometric functions

$$R_+(\rho) = a_0\,{}_1F_1(\eta - \tau, 2\eta + 1; \rho) \qquad R_-(\rho) = b_0\,{}_1F_1(1 + \eta - \tau, 2\eta + 1; \rho). \tag{27}$$

It is shown from equations (17), (25) and (27) that $G_{KE}(\rho)$ and $F_{KE}(\rho)$ can be directly obtained by the combinations of the confluent hypergeometric functions.

We now study the relation between the coefficients a_0 and b_0. Before proceeding to do so, we recall the following recursive relations between the confluent hypergeometric functions [34]

$$\begin{aligned}&\gamma\frac{\mathrm{d}}{\mathrm{d}z}\,{}_1F_1(\alpha, \gamma; z) = \alpha\,{}_1F_1(\alpha + 1, \gamma + 1; z) = \frac{\alpha\gamma}{z}[\gamma\,{}_1F_1(\alpha + 1, \gamma; z) - \gamma\,{}_1F_1(\alpha, \gamma; z)]\\ &\alpha\,{}_1F_1(\alpha + 1, \gamma + 1; z) = (\alpha - \gamma)\,{}_1F_1(\alpha, \gamma + 1; z) + \gamma\,{}_1F_1(\alpha, \gamma; z)\\ &\alpha\,{}_1F_1(\alpha + 1, \gamma; z) = (z + 2\alpha - \gamma)\,{}_1F_1(\alpha, \gamma; z) + (\gamma - \alpha)\,{}_1F_1(\alpha - 1, \gamma; z).\end{aligned} \tag{28}$$

It is shown from equations (21), (27) and (28) that

$$\left(\frac{\eta - \tau}{\rho}a_0 + \frac{\tau' + K}{\rho}b_0\right)\,{}_1F_1(1 + \eta - \tau, 2\eta + 1; \rho) = 0. \tag{29}$$

Since both a_0 and b_0 cannot be vanishing, we obtain

$$b_0 = \frac{\tau - \eta}{\tau' + K}a_0. \tag{30}$$

From equation (17) we thus have

$$\begin{aligned}G_{KE}(\rho) &= N_{KE}\sqrt{M - E}\,\rho^\eta\,\mathrm{e}^{-\rho/2}[(\tau' + K)\,{}_1F_1(\eta - \tau, 2\eta + 1; \rho)\\ &\quad + (\tau - \eta)\,{}_1F_1(1 + \eta - \tau, 2\eta + 1; \rho)]\\ F_{KE}(\rho) &= N_{KE}\sqrt{M + E}\,\rho^\eta\,\mathrm{e}^{-\rho/2}[(\tau' + K)\,{}_1F_1(\eta - \tau, 2\eta + 1; \rho)\\ &\quad - (\tau - \eta)\,{}_1F_1(1 + \eta - \tau, 2\eta + 1; \rho)]\end{aligned} \tag{31}$$

where the normalization factor $N_{KE} = a_0(\tau' + K)^{-1}(2\sqrt{M^2 - E^2})^{-1/2}$ can be determined later.

We now study the eigenvalues of this quantum system. The quantum condition is obtained from the finiteness of the solutions at infinity

$$\tau - \eta = n' = 0, 1, 2, \ldots. \tag{32}$$

When $n' = 0$, $\eta = \tau$ and

$$K^2 = \tau^2 + \xi^2 = (\tau')^2. \tag{33}$$

Therefore, K has to be positive in order to avoid the trivial solution.

Introduce the principal quantum number n

$$n = |K| - (D-3)/2 + n' = |K| - (D-3)/2 + \tau - \eta = l + 1 + n' = 1, 2, \ldots \tag{34}$$

where the principal quantum number n can be equal to 1 only for $K = (D-1)/2$ and equal to other positive integers for both signs of K. The energy E can be obtained from equations (20), (22) and (34)

$$E(n, l, D) = M\frac{\xi}{|\xi|}\left[1 + \frac{\xi^2}{(\sqrt{K^2 - \xi^2} + n - l - 1)^2}\right]^{-1/2}. \tag{35}$$

For a large D, we have

$$E(n, D) \simeq M\frac{\xi}{|\xi|}[1 - 2\xi^2 D^{-2} + 4\xi^2(2n-3)D^{-3} - \cdots] \tag{36}$$

which implies that the energy is independent of l for large D.

For small ξ, we have

$$E(n, l, D) \simeq M\frac{\xi}{|\xi|}\left\{1 - \frac{\xi^2}{2[n + (D-3)/2]^2} - \frac{\xi^4}{2[n + (D-3)/2]^4}\left(\frac{2n + D - 3}{2l + D - 1} - \frac{3}{4}\right)\right\} \tag{37}$$

where the first term on the right-hand side is the rest energy M ($c^2 = 1$ in our conventions), the second one is from the solutions of the Schrödinger equation and the third one is the fine structure energy, which removes the degeneracy between states with the same n.

We now determine the normalization factor N_{KE} from the normalization condition

$$\int \Psi_{KE}^{\dagger}\Psi_{KE}\,\mathrm{d}V = 1. \tag{38}$$

Noting $n' = \tau - \eta$ is a non-negative integer, we can express the confluent hypergeometric function by the associated Laguerre polynomial [34]

$$L_n^{\beta}(\rho) = \frac{\Gamma(\beta + n + 1)}{n!\Gamma(\beta + 1)}\,{}_1F_1(-n, \beta + 1; \rho). \tag{39}$$

$$\int_0^{\infty} \rho^{\beta}\,\mathrm{e}^{-\rho}L_n^{\beta}(\rho)L_m^{\beta}(\rho)\,\mathrm{d}\rho = \frac{\Gamma(n + \beta + 1)}{n!}\delta_{nm}. \tag{40}$$

Through a direct calculation we obtain

$$N_{KE} = \frac{(M^2 - E^2)^{1/4}}{\Gamma(2\eta + 1)}\left[\frac{\Gamma(\tau + \eta + 1)}{2M\tau'(K + \tau')(\tau - \eta)!}\right]^{1/2}. \tag{41}$$

Before ending this section, it should be pointed out that the present approach called the Tricomi equation approach is better than the power series one, which is very complicated in this physical problem, as shown in our previous work [17].

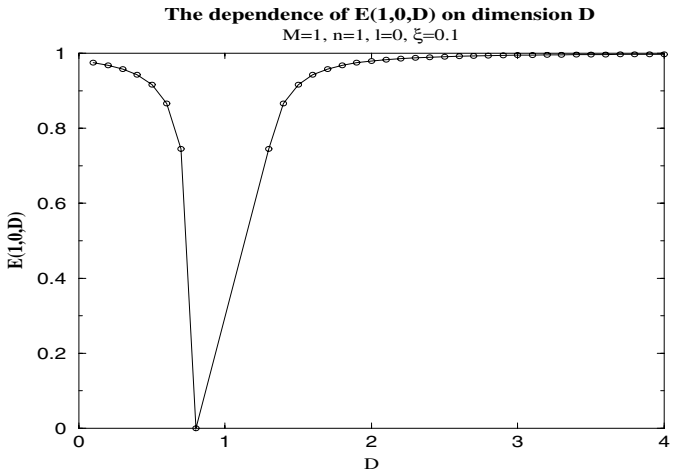

Figure 1. It is shown that the energy $E(1, 0, D)$ decreases with increasing dimension $D \in (0, 0.8]$, but it increases with increasing dimension $D \geqslant 1.2$.

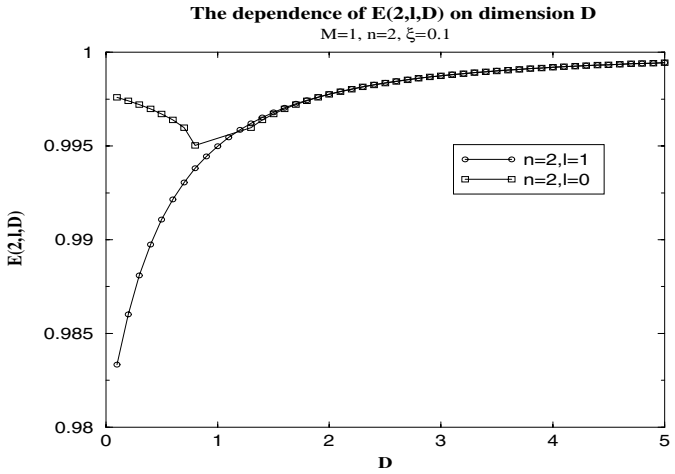

Figure 2. It is shown that the change rule of the energy $E(2, 0, D)$ is very similar to that of $E(1, 0, D)$. The energy $E(2, 1, D)$ increases with increasing dimension D. Especially, it is found that, for $D > 1$, the energy $E(2, 1, D)$ almost overlaps $E(2, 0, D)$.

4. Analysis of the eigenvalues

Recently, Nieto has used the property of *continuous* dimension D to study the bound states of a quantum system with a special potential [21]. In this case, we want to discuss what will happen if we consider the dimension D as a *continuous* variable.

We are now analysing the dependence of the energy $E(n, l, D)$ on the dimension D by figures 1–3 for $n = 1, 2, 3$, respectively. It is shown in figure 1 that the energy $E(1, 0, D)$ decreases as D increases in the region (0, 0.8], but increases as D increases in the region $D \geqslant 1.2$. Figure 2 shows the properties of $E(2, l, D)$ ($l = 1$ or 0). The dependence of $E(2, 0, D)$ on D is similar to that of the energy $E(1, 0, D)$. But the energy $E(2, 1, D)$ increases monotonically as D increases. It is found that, when $D > 1$, the curve of $E(2, 1, D)$ almost

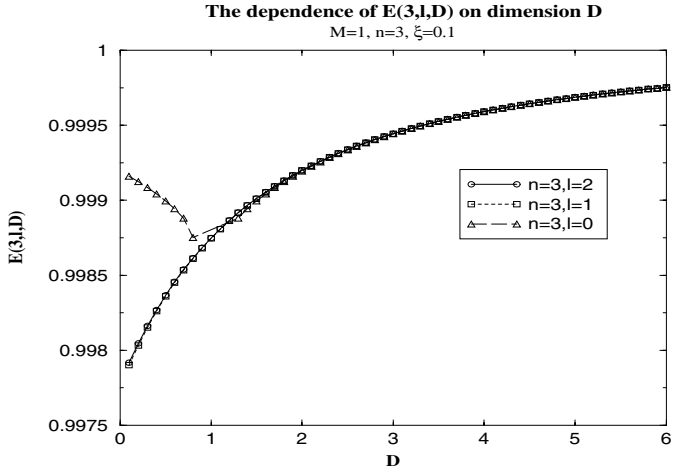

Figure 3. The change rule of $E(3, l, D)$ $(l = 2, 1, 0)$ is similar to that of $E(2, l, D)$ $(l = 1, 0)$. It is found that the energy $E(3, 2, D)$ almost overlaps the energy $E(3, 1, D)$. When $D > 1$ the energies $E(3, l, D)$ $(l = 2, 1, 0)$ almost overlap.

overlaps with that of $E(2, 0, D)$. This kind of phenomenon occurs in $n = 3$ $(l = 2, 1, 0)$ as shown by figure 3. That is to say, the energy $E(3, 2, D)$ almost overlaps with the energy $E(3, 1, D)$. This can be explained well through the power series expansion for $1 = D$ given by equation (36).

5. Discussions of the special case $D = 1$

In this section we will address the reasons why there is an absence of bound states for the one-dimensional case. In fact, a similar physical problem, i.e. the one-dimensional relativistic hydrogen atom, has been studied in some detail by Moss *et al* [18]. However, we have tried to give the reasons why the eigenvalues and eigenfunctions for $D = 1$ do not exist.

To make this problem clear, it is necessary to use a large space to explain the reasons. We want to discuss this interesting problem from two aspects. First we pay more attention to the eigenvalues given in equation (35). Under the case of $D = 1$, which implies the angular momentum $l = 0$, it is found that the eigenvalues can be written as

$$E(n, 0, 1) = M\frac{\xi}{|\xi|}\left[1 + \frac{\xi^2}{(\sqrt{K^2 - \xi^2} + n - 1)^2}\right]^{-1/2}. \tag{42}$$

On considering the variable $K = \pm(2l + D - 1)/2$ defined by equation (12), it is clear that $K = 0$ when $l = 0$ and $D = 1$. Therefore, the factor $\sqrt{K^2 - \xi^2}$ appeared in equation (42) becomes an *imaginary* value regardless of the value of parameter ξ, where $\xi \neq 0$. Therefore, the eigenvalues (42) with the *imaginary* value are not acceptable in physics, that is to say, there is an absence of bound states for $D = 1$. For the case $\xi = 0$, the present work will lose its significance. Not come singly but in pairs, a similar conclusion about the one-dimensional relativistic hydrogen atom has been drawn by Moss *et al* [18], which can be referred to for detailed information.

Second, we want to discuss this problem from another point of view, namely, from the eigenfunctions of $D = 1$. It is known from equation (12) that $K = 0$ when $D = 1$ and $l = 0$.

Therefore, it is shown from equation (21) that

$$\begin{aligned}\Phi'_+(\rho) - \left(\frac{\tau}{\rho} - \frac{1}{2}\right)\Phi_+(\rho) &= -\frac{\tau'}{\rho}\Phi_-(\rho)\\ \Phi'_-(\rho) + \left(\frac{\tau}{\rho} - \frac{1}{2}\right)\Phi_-(\rho) &= \frac{\tau'}{\rho}\Phi_+(\rho)\end{aligned} \tag{43}$$

from which one can obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \left(-\frac{1}{4} + \frac{\tau \pm 1/2}{\rho} + \frac{\xi^2}{\rho^2}\right)\right]\Phi_\pm(\rho) = 0$$

where τ and τ' are given by equation (20). As shown in equations (22) and (23), from the behaviour of the wavefunctions at the origin, it is found that the wavefunctions behave like $\rho^{\mathrm{i}\xi}$, namely, $\eta = \mathrm{i}\xi$ appearing in equations (22)–(23), (25)–(27) and (29)–(31) takes an imaginary value, which is not acceptable for the wavefunctions. Hence, in the case of $D = 1$, it is concluded that there is an absence of bound states from the eigenvalues and eigenfunctions for $D = 1$.

6. Concluding remarks

With the recent interest of higher dimensional field theory, we have studied the $(D + 1)$-dimensional Dirac equation with the Coulomb potential following the Tricomi equation approach, which is different from the method used in our previous work [17, 24]. The eigenfunctions are analytically obtained and expressed by the confluent hypergeometric function. The eigenvalues as well as their fine structure energy for this system are also studied. It is worth pointing out that when the Coulomb potential is not very strong, an attractive potential leads to bound states with positive energies, and a repulsive potential leads to those with negative energies. The present work pays more attention to the dependences of the energy levels $E(n, l, D)$ on the *continuous* dimension D as proposed in [21]. For definiteness, we assume that $\xi = 0.1$ and $M = 1$. For the energy $E(n, l, D)$, we obtain the following property: the $E(n, 0, D)$ first decreases with increasing dimension D and then increases with increasing D. The energy $E(n, l, D)$ with $l > 0$ is almost independent of the quantum number l for large D. For the energy levels $E(n, 0, D)$ there is an absence of bound states for $D = 1$. This can be explained well from the factor $\sqrt{K^2 - \xi^2}$ appearing in equation (35), which in this case has an imaginary value if we consider the definition of K given in equation (12). This is not acceptable in physics. Also this can be explained from the eigenfunctions for $D = 1$, which also take an imaginary value as discussed above. The solutions obtained in this work coincide with those in $(3 + 1)$-dimensional [16] and $(2 + 1)$-dimensional [17] spacetime.

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